Denotational Semantics

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Lectures for Part II CST 2026/2027

Recap of Lecture 2

We recalled the definitions of:

- partially ordered set (poset)
- least elements and least upper bounds
- chain-complete posets (cpo) and domains

and we proved basic property of least upper bounds:

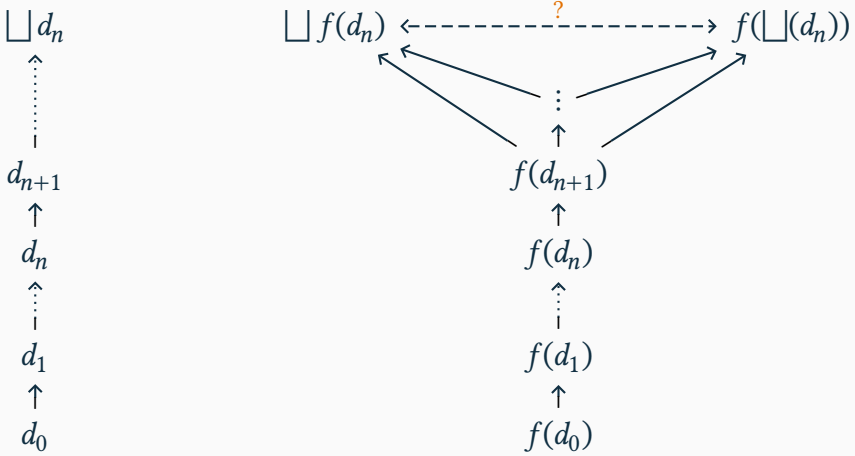
- uniqueness
- monotonicity
- diagonalisation for doubly indexed sequences

Domains and Fixed Points

Continuous functions

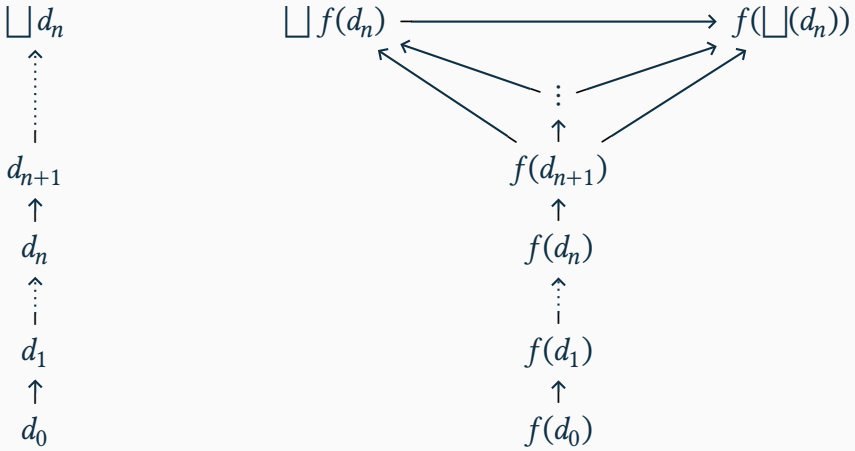
Monotone functions and lubs

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Continuity and strictness

Given two cpos D and E, a function $f : D \to E$ is called continuous if

- it is monotone, and
- it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, we have

$$\bigsqcup f(d_n) = f(\bigsqcup d_n) \qquad (\sqsubseteq \text{ is automatic})$$

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A continuous function $f : D \to E$ between domains is **strict** when $f(\bot_D) = \bot_E$.

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Typical non-continuous function: “is a sequence the constant 0”? $(\mathbb{N} \rightharpoonup \mathbb{B}) \rightharpoonup \mathbb{B}$

$$0 \quad 0 \quad \bot \quad \ldots \qquad \mapsto \bot$$
$$0 \quad 0 \quad 0 \quad 0 \quad 1 \quad \ldots \qquad \mapsto 1$$

$$0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \overline{0} \qquad \mapsto 0$$

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Later in the course: we show the thesis… by giving a denotational semantics.

Domains and Fixed Points

Fixed points

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It is (uniquely) specified by the two properties:

$$\textsc{lfp-fix}\ \frac{}{f(\mathrm{fix}(f)) \sqsubseteq \mathrm{fix}(f)} \qquad \textsc{lfp-least}\ \frac{f(d) \sqsubseteq d}{\mathrm{fix}(f) \sqsubseteq d}$$

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- $\mathrm{fix}(f)$ is a pre-fixed point
- To prove $\mathrm{fix}(f) \sqsubseteq d$, it is enough to show $f(d) \sqsubseteq d$.

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Theorem

Let $f : D \to D$ be a continuous function on a domain D. Then f possesses a least pre-fixed point, given by

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We need to check that:

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Question: What is $\mathrm{fix}(f)$ when f is strict?

Constructions on Domains

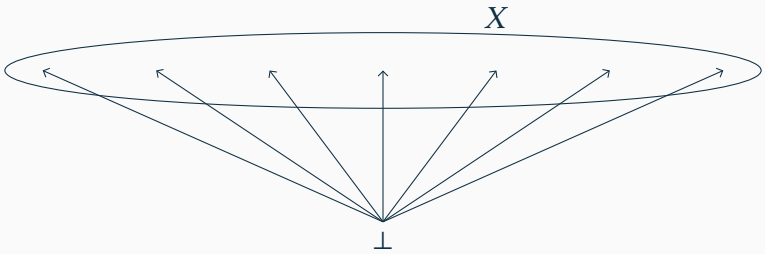
Constructions on Domains

Flat domains

Flat domain on X

The flat domain on a set X is defined by:

- its underlying set $X \uplus \{\bot\}$ *i.e.* X extended with a new element $\bot$;
- $x \sqsubseteq x'$ if either $x = \bot$ or $x = x'$.



Flat domain lifting

Let $f : X \rightharpoonup Y$ be a partial function between two sets. Then

$$
\begin{array}{rccl}
f_\perp : & X_\perp & \to & Y_\perp \\
 & d & \mapsto & \begin{cases} f(d) & \text{if } d \in X \text{ and } f \text{ is defined at } d \\ \perp & \text{if } d \in X \text{ and } f \text{ is not defined at } d \\ \perp & \text{if } d = \perp \end{cases}
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defines a strict continuous function.

Flat domain lifting

Let $f : X \rightharpoonup Y$ be a partial function between two sets. Then

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$$

defines a strict continuous function.

Moreover, every strict continuous function $X_\bot \to Y_\bot$ arises this way!

Constructions on Domains

Binary products

Binary product of posets

The product of two posets $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ has underlying set

$$D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \wedge d_2 \in D_2\}$$

and partial order $\sqsubseteq$ defined componentwise:

$$(d_1, d_2) \sqsubseteq (d_1', d_2') \overset{\text{def}}{\Leftrightarrow} d_1 \sqsubseteq_1 d_1' \wedge d_2 \sqsubseteq_2 d_2'$$

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$$\text{PO}\times \frac{d_1 \sqsubseteq_1 d_1' \qquad d_2 \sqsubseteq_2 d_2'}{(d_1, d_2) \sqsubseteq (d_1', d_2')}$$

Binary product of cpos and domains

Lubs of chains are computed componentwise:

$$\bigsqcup_n (d_{1,n}, d_{2,n}) = \left(\bigsqcup_i d_{1,i}, \bigsqcup_j d_{2,j} \right).$$

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Therefore, products of cpos are cpos, and products of domains are domains.

Functions of two arguments

A function $f : (D \times E) \to F$ is monotone exaclty when it is monotone in each argument:

$$\forall d, d' \in D, e \in E. d \sqsubseteq d' \Rightarrow f(d, e) \sqsubseteq f(d', e)$$
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It is continuous if and only if it preserves lubs in each argument separately:

$$f(\bigsqcup_m d_m\,,\, e) = \bigsqcup_m f(d_m, e) \qquad f(d\,,\, \bigsqcup_n e_n) = \bigsqcup_n f(d, e_n).$$

Derived rules for functions of two arguments

$$\text{MON}\times\ \frac{f \text{ monotone} \qquad x \sqsubseteq x' \qquad y \sqsubseteq y'}{f(x,y) \sqsubseteq f(x',y')}$$

$$f\left(\bigsqcup_m x_m, \bigsqcup_n y_n\right) = \bigsqcup_m \bigsqcup_n f(x_m, y_n) = \bigsqcup_k f(x_k, y_k)$$

Projections and pairing

Let D_1 and D_2 be cpos (domains). The projections

$$\begin{array}{rccc} \pi_1 : & D_1 \times D_2 & \to & D_1 \\ & (d_1, d_2) & \mapsto & d_1 \end{array} \qquad \begin{array}{rccc} \pi_2 : & D_1 \times D_2 & \to & D_2 \\ & (d_1, d_2) & \mapsto & d_2 \end{array}$$

are (strict) continuous functions.

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are (strict) continuous functions.

If $f_1 : D \to D_1$ and $f_2 : D \to D_2$ are (strict) continuous functions from a cpo (domain) D, then their pairing:

$$\begin{array}{rccc} \langle f_1, f_2 \rangle : & D & \to & D_1 \times D_2 \\ & d & \mapsto & (f_1(d), f_2(d)) \end{array}$$

is (strict) continuous.

Application: Domain Conditional

For any domain D, the conditional function

$$\begin{array}{rrcl} \text{if}: & \mathbb{B}_\perp \times (D \times D) & \to & D \\ & (x, d) & \mapsto & \begin{cases} \pi_1(d) & \text{if } x = \text{true} \\ \pi_2(d) & \text{if } x = \text{false} \\ \perp_D & \text{if } x = \perp_{\mathbb{B}} \end{cases} \end{array}$$

is (strict) continuous.

Constructions on Domains

General products

General product

The (cartesian) product of a family of sets $(X_i)_{i \in I}$ indexed by a set I is the set

$$\prod_{i \in I} X_i = \left\{ p \colon I \to \bigcup_{i \in I} X_i \mid \forall i \in I \,.\, p(i) \in X_i \right\}$$

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We can think of its elements equivalently either as

- I-indexed tuples: $(\dots, x_i, \dots)_{i \in I}$ such that $x_i \in X_i$;
- heterogeneous functions: p defined on I such that $p(i) \in X_i$.

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Special case: $\prod_{i \in \mathbb{B}} D_i$ is canonicall bijective to $D_{\text{true}} \times D_{\text{false}}$.

It is equipped with projection functions (for any $i \in I$) and pairing:

$$\pi_i : \left(\prod_{i \in I} X_i \right) \to X_i \qquad \langle - \rangle_{i \in I} : \prod_{i \in I} (X \to X_i) \to \left(X \to \prod_{i \in I} X_i \right)$$

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- underlying set $\prod_{i \in I} D_i$;
- componentwise order

$$p \sqsubseteq p' \overset{\text{def}}{\Leftrightarrow} \forall i \in I.\, p(i) \sqsubseteq_i p'(i).$$

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The projections are (strict) continuous and the pairing of (strict) continuous functions is (strict) continuous.

Separated sum of domains

The separated sum of a family of domains $(D_i, \sqsubseteq_i)_{i \in I}$ has as carrier set:

$$\sum_{i \in I} D_i = \{\bot\} \cup \bigcup_{i \in I} \{i\} \times D_i$$

- its elements are of the form (i, d) where $i \in I$ and $d \in D_i$
- there is a new bottom element $\bot$.

Its order is given by:

- $\bot \sqsubseteq d$ for any $d \in \sum_{i \in I} D_i$
- $(i, d) \sqsubseteq (j, d')$ if $i = j$ and $d \sqsubseteq_{D_i} d'$

Injections and Case

The inclusion functions for every $i \in I$:

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The case function:

$$[-]_{i \in I} : \prod_{i \in I} (D_i \to D) \to \left(\sum_{i \in I} D_i \to D \right)$$

$$(f_i)_{i \in I} \mapsto \begin{cases} \bot & \mapsto \bot_D \\ (i, d) & \mapsto f_i(d) \end{cases}$$

takes continuous functions to a strict continuous function.